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Is the preference of the majority representative ?[☆]

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ABSTRACT

We show that a majoritarian relation is, among all conceivable binary relations, the most representative of the profile of preferences from which it emanates. We define "the most representative" to mean that it minimizes the sum of distances between itself and the preferences in the profile for a given distance function. We identify a necessary and sufficient condition for such a distance to always be minimized by a majoritarian relation. This condition requires the distance to be additive with respect to a plausible notion of compromise between preferences. The well-known Kemeny distance does satisfy this property, along with many others. All distances that satisfy this property can be written as a sum of strictly positive weights assigned to the ordered pairs of alternatives by which any two preferences differ.

1. Introduction

The "preference of the majority" is indisputably one of the most widely used and discussed social preference. However, the normative justifications of the "majoritarian" way of aggregating individual preferences are surprisingly thin. One important justification has been provided by May (1952), who proves that when there are only two alternatives to be compared, the majority rule is the only mapping of individual preferences into a social ranking that is *complete*, *anonymous*, *neutral* and *positively responsive*. A well-known (at least since Condorcet in the late XVIIIth century) limitation of the majority rule is its failure to satisfy transitivity. This limitation is obviously not addressed by May (1952) who restricts his analysis to the two-alternatives case. In the discussion of his impossibility theorem, Arrow (1963) (p. 101), recognizes that the generalization of May's result to more than two alternatives is not easy. Papers who have proposed such a generalization include Dasgupta and Maskin (2008) and Horan et al. (2019). However, they have done so in the case where the individual preferences are so restricted that the majority rule is transitive or, at least, admits a maximal element (called a Condorcet winner).

An alternative way of justifying the preference of the majority would be of course to combine results in May (1952) and Arrow (1950) through the well-known axiom of *binary independence from irrelevant alternatives*, which requires the social ranking of any two alternatives to depend only upon the individuals' ranking of these two alternatives. One could then justify the preference of the majority as being the only mapping of individual preferences into a social ranking that is *complete*, *anonymous*, *neutral*, *positively responsive*, and *binarily independent of irrelevant alternatives*. However, as recognized by May (1952) himself, the appeal of the binary independence of irrelevant alternatives and neutrality axioms is not clear.

In this paper, we therefore examine an alternative justification for the majority rule. Specifically, we show that the preference of the majority qualifies as being *representative* of the collection of preferences from which it emanates. The notion of representativeness on which our argument rides is that underlying the choice of several measures of "central tendency" in classical statistics. A common justification indeed for the *mean* of a set of numbers as a "representative statics" for these numbers is that the mean minimizes the sum of the squares of the differences between itself and the represented numbers. Similarly, the *median* of a set of numbers – another widely used measure of "central tendency" – is commonly justified by the fact that it minimizes the sum of the absolute values of those same differences, while the *mode* minimizes the degenerate distance between numbers that is 1 if the numbers differ and 0 if they do not. In a similar spirit, it is common in *regression analysis* to fit a cloud of points indicating the values taken by a "dependent" variable and a collection of "independent" ones by a specific function whose parameters are "estimated" by minimizing the sum of the (squares

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of) the discrepancies between the predicted and observed values of the dependent variables. The parametric curve estimated in this fashion is commonly portrayed as "representative" of the cloud of points.

We show herein that the "preference of the majority" is representative in an analogous fashion of the collection of preferences from which it emanates. We specifically show that the preference of the majority minimizes the sum of distances between itself and the preferences for a significant class of distance functions over these preferences which we characterize. We indeed identify the property that any distance on preferences – described as binary relations – must satisfy in order to be minimized by a majoritarian preference. This property happens to be that of *additivity* of the distance with respect to any three binary relations such that one of them is a Paretian aggregation of the two others. It is common to refer to a preference which is a Paretian aggregation of two others preferences as being "in-between" those two. A distance is additive in this sense if for any two preferences, the sum of the distances between each of the two preferences and a Paretian aggregation of them is always equal to the distance between these two preferences.

Our analysis can thus be seen as a generalization of a literature that discusses the representativeness of the majoritarian preference – in the sense of distance minimization – with respect to the specific *Kemeny* (or Kendall) distance initially characterized by Kemeny (1959) and Kemeny and Snell (1962) for linear preferences, and significantly generalized to weak and even non-transitive preferences by Bogart (1973, 1975). It has been known indeed since at least (Barbut, 1980) that the preference of the majority minimizes the sum of pairwise disagreements between itself and all preferences from which it emanates (see Monjardet (2005) for a good synthesis). While this Kemeny distance minimization property of the preference of the majority is very often appealed to in contexts where the preference of the majority is transitive (see for example Demange (2012)), Barbut (1980) has indicated that nothing in the argument was depending upon transitivity. The literature has also established that the majoritarian preference can be seen as the 'median' preference in a metric space over preferences in which the metric is the Kemeny distance. For example, Young and Levenglick (1978) have characterized in this fashion all Condorcet consistent rules.

The current paper extends the results on the representativeness of the majority by showing that it holds for the significantly larger class of all distances that are additive – in the sense above – between any three preferences such that one is between the other two. This property of additivity has been used as a primitive axiom, along with others, in all the characterizations of the Kemeny distance that we are aware of (in particular those of Kemeny (1959), Kemeny and Snell (1962), Bogart (1973, 1975) and, more recently, Bossert et al. (2016)). We show in this paper that this property of additivity characterizes in fact the much larger class of distances between any two preferences that can be written as a sum of more elementary distances between the pairs of alternatives by which the two preferences differ. While the Kemeny distance is one such distance – which assumes that any two distinct alternatives have a distance of 1 – there are many others that allow different pairs of alternatives to have different positive distances. All such additive distances, and only them, happen to be minimized by the preference of the majority.

The plan for the remainder of the paper is as follows. The next section introduces the notation and the model and provides the main results while Section 3 concludes.

2. The model and the results

We start our analysis by recalling some terminology and notation pertaining to binary relations. By a *binary relation* R on a finite set X, we mean a subset of $X \times X$. For any binary relation R on X, we define its *symmetric* factor $I(R)$ by $(x, y) \in I(R) \iff [(x, y) \in R \text{ and } (y, x) \in R]$ and its *asymmetric* factor $P(R)$ by $(x, y) \in P(R) \iff [(x, y) \in R \text{ and } (y, x) \notin R]$. A binary relation R is *asymmetric* when it coincides with its asymmetric factor. A binary relation R on X is:

(i) *reflexive* if $(x, x) \in R$ for every $x \in X$.

(ii) *transitive* if, for any x, y and $z \in X$, $(x, z) \in R$ always follows $(x, y) \in R$ and $(y, z) \in R$.

(iii) *complete* if $\{(x, y), (y, x)\} \cap R \neq \varnothing$ for every distinct $x, y \in X$.

While binary relations often considered in social choice theory are taken to be reflexive, transitive and complete, these two latter assumptions will play no major role in the current analysis. Indeed, the results that will be established about the representativeness of the majoritarian binary relation – that is itself often intransitive as we all know – are valid also when the binary relations over which the majority is defined are neither transitive nor complete. However, the requirement that these binary relations be reflexive will play some role in the arguments.[1] We accordingly denote by $\mathcal{R}$ the set of all reflexive binary relations. Finally, for any two binary relations R and R', we denote by $R \vartriangle R'$ their symmetric set difference defined by $R \vartriangle R' = (R \cup R') \backslash (R \cap R')$.

We begin by discussing the notion of a compromise between two binary relations. The cornerstone of the compromise's idea is that of a (Pareto) respect for unanimity. This idea underlies the following notion of *betweenness* between two binary relations.

Definition 1. For any two binary relations R and R'' on X, we say that the binary relation $R' \in \mathcal{R}$ is between R and R'' if only if $(R \cap R'') \subseteq R' \subseteq (R \cup R'')$.

In words, R' is between R and R'' if R' always agrees with the unanimity of R and R'' – when the latter occurs – and, somewhat conversely, never expresses a preference for one alternative over the other if this preference is not also expressed by either R or R''. We observe trivially that this notion of betweenness is symmetric: R' is indeed between R and R'' if and only if it is between R'' and R. It turns out that an alternative – but actually equivalent – definition of betweenness can be formulated for *complete* binary relations. This equivalent definition makes, in our view, the notion of betweenness underlying Definition 1 even more intuitive. We formulate this equivalent definition in the following lemma proved, like all formal results in the paper, in the Appendix.

Lemma 1. *Let R, R' and R'' be three complete binary relations on X. Then R' is between R and R'' as per Definition 1 if and only if it satisfies:*

(i) $(x, y) \in R$ and $(x, y) \in R'' \Longrightarrow (x, y) \in R'$ and,

(ii) $(x, y) \in P(R)$ and $(x, y) \in P(R'') \Longrightarrow (x, y) \in P(R')$.

Hence, as stated in Lemma 1, a (complete) binary relation is between two others if and only if it results from a Paretian aggregation of those two binary relations.[2] For any two binary relations

1 The requirement that binary relations be reflexive is actually not crucial for the analysis. We could just equally well assume the binary relations R considered herein to be irreflexive and, therefore, be such that $(x, x) \notin R$ for all alternatives $x \in X$. However, it is important that the reflexivity/irreflexivity status of an alternative be the same for all alternatives considered. We would have some difficulties in handling (weird) binary relations R that would consider that $(x, x) \in R$ for some x and $(y, y) \notin R$ for some other y.

2 The Paretian aggregation underlying Definition 1 satisfies what is known in the social choice literature (see e.g. Suzumura (2001)) as the weak Pareto principle. Other authors, such as Grandmont (1978), have defined betweenness in terms of a Paretian aggregation satisfying the strong Pareto principle.

R and R'', we let $\mathcal{B}(R, R'') = \{R' \subset X \times X : R'$ is between R and $R''\}$. Since, for any R and R'', both R and R'' are (trivially) between R and R'', the set $\mathcal{B}(R, R'')$ is never empty. We now formally define the *majoritarian* binary relation associated with any given profile of binary relations.[3]

Definition 2. Given a profile of n binary relations $(R_1, \ldots, R_n) \in \mathcal{R}^n$ for some integer $n \geq 2$, we say that the binary relation *R on X* is majoritarian for $(R_1, \ldots, R_n)$ *if* $R \in \mathcal{B}(R^w(R_1, \ldots, R_n), R^s(R_1, \ldots, R_n))$ where the weak and the strict majority relations $R^w(R_1, \ldots, R_n)$ and $R^s(R_1, \ldots, R_n)$ *associated to* $(R_1, \ldots, R_n)$ *are defined by:*
$R^w(R_1, \ldots, R_n) = \{(x, y) \in X \times X : \#\{i : (x, y) \in R_i\} \geq n/2\}$ and
$R^s(R_1, \ldots, R_n) = \{(x, y) \in X \times X : \#\{i : (x, y) \in R_i\} > n/2\}$.

The strict majority relation $R^s(R_1, \ldots, R_n)$ has been called the *Condorcet relation* by Barbut (1980). A binary relation R is therefore defined to be majoritarian if it is between the Condorcet relation $R^s(R_1, \ldots, R_n)$ and the weak majority relation $R^w(R_1, \ldots, R_n)$. Observe that when n is odd, then $R^s(R_1, \ldots, R_n)$ is the unique majoritarian relation associated to $(R_1, \ldots, R_n)$. Observe also that when n is odd, this unique majoritarian relation $R^s(R_1, \ldots, R_n)$ will be complete if the binary relations $R_1, \ldots, R_n$ are all complete. However, when n is even, there will typically be many majoritarian relations, some of them possibly incomplete. For example, if one considers the profile of two binary relations R_1 and R_2 where, for two distinct alternatives x and y, $(x, y) \in I(R_1)$ and $(x, y) \in P(R_2)$, then one observes that $(x, y) \in P(R^s(R_1, R_2))$ and $(x, y) \in I(R^w(R_1, R_2))$ are two different majoritarian binary relations (when restricted to the set $\{x, y\}$). Moreover, one can observe that if all binary relations in the profile $(R_1, \ldots, R_n)$ are complete, then there will always exist at least one complete majoritarian relation (irrespective of whether n is odd or even).

We also record for further reference the following obvious (and therefore unproved) remark that results from the definition of a majoritarian binary relation (given the definition of betweenness).

Remark 1. A *binary relation R is majoritarian with respect to the profile of* n binary relations $(R_1, \ldots, R_n)$ (for some integer $n \geq 2$) if and only if it satisfies, for *every* x and $y \in X$, $(x, y) \in R \Longrightarrow \#\{i : (x, y) \in R_i\} \geq n/2$ and $(x, y) \notin R \Longrightarrow \#\{i : (x, y) \in R_i\} \leq n/2$.

The main contribution of the paper is to characterize any majoritarian binary relation over some profile as a minimizer of the sum of the pairwise *distances* between itself and the binary relations of the profile for some distance function. As it turns out, the distance-minimizing property of a majoritarian binary relation depends crucially upon a property of the distance that we refer to as "between-additivity".

We start by recalling the definitions of semi-distance and distance functions as applied to any set S of objects (which could, of course, be binary relations).

Definition 3 (*Semi-Distance and Distance*)**.** Given a set S, we call **semi-distance** on S any function $d : S \times S \to \mathbb{R}_+$ that satisfies, for any a and $b \in S$:

(i) Zero at Identity Only: $d(a, b) = 0$ if and only if $a = b$.

(ii) Symmetry: $d(a, b) = d(b, a)$.

Moreover, we call **distance** any function that satisfies, along with (i) and (ii):

(iii) Triangle Inequality: $d(a, c) \leq d(a, b) + d(b, c)$ for all a, b and $c \in S$.

[3] We are grateful to a referee for suggesting this formal definition of a majoritarian relation.

We now introduce as follows the property of between-additivity of a semi-distance when applied to binary relations.

Definition 4. A function $d : \mathcal{R} \times \mathcal{R} \to \mathbb{R}_+$ is **between-additive** if it satisfies $d(R_1, R_3) = d(R_1, R_2) + d(R_2, R_3)$ for every R_1, R_2 and $R_3 \in \mathcal{R}$ such that $R_2 \in \mathcal{B}(R_1, R_3)$.

The property of between-additivity requires the function d to be "additive" with respect to any combination of two binary relations taken from three binary relations that are connected by a betweenness relation. This property, when imposed on a semi-distance defined on a set of binary relations, is quite strong since it implies, among other things, that the semi-distance be actually a distance and, therefore, satisfies the Triangle Inequality. We state this fact formally in the following Lemma (see also Lemma 1 in Bossert et al. (2016)), that we prove in the Appendix.[4]

Lemma 2. *Let* $d : \mathcal{R} \times \mathcal{R} \to \mathbb{R}_+$ *be a between-additive semi-distance on* $\mathcal{R}$*. Then d satisfies Triangle Inequality and is therefore a between-additive distance.*

The Triangle Inequality is not the only implication of the property of between-additivity when applied to semi-distance functions. As it turns out, any between-additive semi-distance – or distance thanks to Lemma 2 – between two binary relations happens to be a sum of strictly positive weights assigned to the ordered pairs of distinct alternatives by which the two binary relations differ. We state this implication formally in the following proposition (proved in the Appendix).[5]

Proposition 1. *A semi-distance* $d : \mathcal{R} \times \mathcal{R} \to \mathbb{R}_+$ *is between-additive if and only if there exists a function* $\delta : X \times X \to \mathbb{R}_+$ *such that for any two binary relations* R_1 *and* R_2 *one has:*

$$d(R_1, R_2) = \sum_{(x,y) \in R_1 \Delta R_2} \delta(x, y)$$

A well-known member of the class of between-additive (semi) distances is the *Kemeny distance*, denoted d^K, for which the function δ^K of Proposition 1 is the discrete distance function defined by:

$$\delta^K(x, y) = 1 \text{ if } x \neq y$$
$$= 0 \text{ otherwise}$$

Along with other axioms, the property of between-additivity has often been used in the literature to characterize the Kemeny distance (see e.g. Kemeny (1959), Kemeny and Snell (1962), Bogart (1973, 1975), Can and Storcken (2018) and Bossert et al. (2016)). Proposition 1 makes clear that the Kemeny distance is only one (very) specific member of the class of between-additive distances. We also find worth emphasizing that the weighting function δ does not need to be a semi-distance. Contrary to what is the case for the Kemeny distance, the pairs (x, y) and (y, x) may indeed be weighted differently in their contribution to the distance between two binary relations.

The following example provides a well-known example of a distance between binary relations that is not between-additive.

[4] Bossert et al. (2016) present their unproved Lemma 1 as "an immediate consequence of Theorem 3 of Can and Storcken (2015)". Yet, to the very best of our understanding, Theorem 3 in Can and Storcken (2015) deals with distances or dissimilarity functions defined over orderings. Our approach here applies to any binary relation. We therefore find useful to provide a proof of the result for this general case as well.

[5] Here again, we are grateful to a referee for suggesting this proposition.

Example 1. The Spearman (1904) distance (see Monjardet (1998) for a comparison of this distance with that of Kemeny) is denoted d^S and is defined by $d^S(R^1, R^2) = [\sum_{x \in X}(r^1(x) - r^2(x))^2]^{1/2}$ where, for $i = 1, 2$, $r^i(x)$ denoted the rank of alternative x in the binary relation R^i defined by:

$$r^i(x) = 1 + \#\{a \in X : (a, x) \in P(R_i)\}$$

It is readily seen that the Spearman distance is not between-additive. For example if we take $X = \{a, b, c\}$ and the binary relations R^1, R^2 and R^3 defined by:

$$R^1 = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\},$$
$$R^2 = \{(a, a), (b, b), (c, c), (b, c), (b, a), (a, c)\} \text{ and}$$
$$R^3 = \{(a, a), (b, b), (c, c), (c, b), (b, a), (c, a)\}$$

it is clear that $R^2 \in \mathcal{B}(R^1, R^3)$. However:

$$d^S(R^1, R^3) = 2\sqrt{2} < d^S(R^1, R^2) + d^S(R^2, R^3) = (1 + \sqrt{3})\sqrt{2}$$

We now turn to the two main results of this paper. The first one states that a binary relation minimizes the sum of semi-distances between itself and a collection of binary relations for some between-additive semi-distance function if and only if the binary relation is majoritarian with respect to the considered collection of binary relations. We state formally this result as follows.

Theorem 1. *Let $d : \mathcal{R} \times \mathcal{R} \to \mathbb{R}_+$ be a between-additive semi-distance (or distance thanks to Lemma 2) function and, for some integer n, let $(R_1, \ldots, R_n)$ be a profile of reflexive binary relations. Then the reflexive binary relation R^* satisfies the inequality:*

$$\sum_{i=1}^{n} d(R_i, R^*) \leq \sum_{i=1}^{n} d(R_i, R) \quad \forall R \in \mathcal{R} \tag{1}$$

if and only if R^* is majoritarian for $(R_1, \ldots, R_n)$.

Remark 2. While the statements of Proposition 1 and Theorem 1 involve the larger set $\mathcal{R}$ of all reflexive binary relations, Theorem 1 can actually be proved (albeit differently) even if the domain of the distance function was limited to complete and reflexive binary relations.[6] In the same vein, Theorem 1 would remain valid if the binary relations $(R_1, \ldots, R_n)$ with respect to which the majoritarian relation is defined were assumed to be complete and transitive.

The next theorem characterizes, somewhat dually, the property of between-additivity of a *distance* as being essential for the ability of a majoritarian binary relation to be representative in the sense of aggregate distance minimization. Specifically, we prove that if a majoritarian binary relation for a given profile of binary relations is to be distance-minimizing with respect to this profile for some distance function, then the distance function *must* be between-additive.

Theorem 2. *Suppose $d : \mathcal{R} \times \mathcal{R} \to \mathbb{R}_+$ is a distance function such that, for every profile $(R_1, \ldots, R_n) \in \mathcal{R}^n$ for some $n \geq 2$, a majoritarian binary relation R^* for this profile satisfies the inequality:*

$$\sum_{i=1}^{n} d(R_i, R^*) \leq \sum_{i=1}^{n} d(R_i, R) \;\; \forall R \in \mathcal{R} \tag{2}$$

Then d is between-additive.

6 The proof of this is available from the authors upon request.

Remark 3. The necessity of the between-additivity of the distance minimized by a majoritarian binary relation established by Theorem 2 rides crucially upon the Triangle Inequality.

Remark 4. The proof of Theorem 2 actually establishes a stronger result than what is stated. Indeed, the proof establishes that for any given even integer n or "large enough" odd integer n, any violation of between-additivity of a distance will give rise to a profile of exactly n binary relations with respect to which a majoritarian binary relation is not distance minimizing.

3. Conclusion

This paper has provided what we believe to be a significant generalization of a (relatively) little known argument in favour of the "preference of the majority" as a rule for collective decision making. We have shown, in effect, that the preference of the majority is representative of the collection of preferences (reflexive binary relations) from which it emanates in the sense of minimizing the sum of (semi)-distances between those binary relations and itself for a reasonably general class of semi-distance functions. We identified the largest class of distances between binary relations that are minimized by the "preference of the majority". This class consists of all distances that are between-additive. The highly specific Kemeny distance, whose minimization by the preference of the majority has been known for quite some time, is of course one member of this class, among many others.

Indeed, as shown in Proposition 1, all semi-distances between binary relations that can be written as an additive combination of more primitive weights assigned to the ordered pairs of alternatives by which the two binary relations differ are minimized by the majoritarian binary relation. The weights assigned to each of the ordered pairs by the semi-distance may be interpreted as reflecting some underlying "objective dissimilarity" between the alternatives that may vary among them, and that may not even be symmetric (the weight assigned to (x, y) may differ from the weight assigned to (y, x)). We find remarkable that majority minimizes even such additive distances that do not assume – as the Kemeny distance does – that all pairs of alternatives have the same dissimilarity. However, the between-additivity of the distances between binary relations happens to be crucial for those distances to be minimized in the aggregate by the "preference of the majority". For, as shown in Theorem 2, any violation of this property enables one, for any population of even size, and "almost any" population of odd size, to construct a profile of binary relations with respect to which majority is not representative in the sense of aggregate distance minimization.

These results leave open at least two avenues for future research. One of them would be to look for alternative binary relations that could qualify as representative in the sense of distance minimization for different families of distances between preferences. While significant, the class of between-additive distances is rather special. Looking at other family of possible distances between binary relations that would be minimized in the aggregate by other preferences than majoritarian strikes us as a worthy topic of investigation. Another avenue for future research would be to identify the properties of a primitive ordinal notion of dissimilarity between preferences that are necessary and sufficient for admitting a numerical representation taking the form of a between-additive distance function. While we can identify some axiomatic properties of an ordinal notion of similarity between preferences that are necessary for admitting such a between-additive numerical representation, the precise identification of simple necessary and sufficient axioms remains to be done.

Appendix. Proofs

A.1. Lemma 1

For one direction of the implication (that does not actually require completeness), let R, R' and R'' be three binary relations on X such that R' is between R and R'' as per Definition 1. Since $(R \cap R'') \subseteq R'$, condition (i) of the Lemma follows. Assume now that x and y are two alternatives such that $(x, y) \in P(R)$ and $(x, y) \in P(R'')$. From the definition of the asymmetric factor of a binary relation, one has $(x, y) \in R$ and $(x, y) \in R''$ and, since $(R \cap R'') \subseteq R'$, one must have $(x, y) \in R'$. We now show that $(y, x) \notin R'$. Suppose to the contrary that $(y, x) \in R'$. Since $R' \subseteq (R \cup R'')$, one must have either $(y, x) \in R$ or $(y, x) \in R''$. But neither of these statements is consistent with the fact that both $(x, y) \in P(R)$ and $(x, y) \in P(R'')$ hold.

For the other direction, assume that R, R' and R'' are three complete binary relations on X for which Statements (i) and (ii) of the lemma hold. Statement (i) clearly implies that $(R\cap R'') \subseteq R'$. Consider now any two alternatives x and y in X such that neither $(x, y) \in R$ nor $(x, y) \in R''$ is true. We wish to show that $(x, y) \in R'$ does not hold. To see this, we observe that, since R and R'' are complete, the fact that neither $(x, y) \in R$ nor $(x, y) \in R''$ is true implies that $(y, x) \in P(R)$ and $(y, x) \in P(R'')$. By statement (ii) of the lemma, this implies that $(y, x) \in P(R')$, which implies in turn, from the very definition of the asymmetric factor of a binary relation, that $(x, y) \notin R'$, as required.

A.2. Lemma 2

By contradiction, suppose that $d : \mathcal{R} \times \mathcal{R} \to \mathbb{R}_+$ is a between-additive semi-distance that violates the Triangle Inequality. This means that there are binary relations R_1, R_2 and $R_3 \in \mathcal{R}$ such that:

$$d(R_1, R_3) > d(R_1, R_2) + d(R_2, R_3) \tag{3}$$

It is clear from this inequality and Definition 4 that $R_2 \notin \mathcal{B}(R_1, R_3)$. We now establish that inequality (3) applied to a between-additive semi-distance also rules out the possibility that $R_1 \in \mathcal{B}(R_2, R_3)$ or that $R_3 \in \mathcal{B}(R_1, R_2)$. Assume indeed that $R_1 \in \mathcal{B}(R_2, R_3)$. By between-additivity (Definition 4), this entails that:

$$d(R_2, R_3) = d(R_2, R_1) + d(R_1, R_3)$$

and, after substituting into the right hand side of inequality (3):

$$\begin{aligned} d(R_1, R_3) &> d(R_1, R_2) + d(R_2, R_1) + d(R_1, R_3) \\ &= 2d(R_1, R_2) + d(R_1, R_3) \text{ (by symmetry)} \\ \iff 0 &> 2d(R_1, R_2) \end{aligned}$$

a contradiction of $d(R_1, R_2) \in \mathbb{R}_+$. Hence $R_1 \notin \mathcal{B}(R_2, R_3)$. An analogous argument leads to the conclusion that $R_3 \notin \mathcal{B}(R_1, R_2)$.

Consider then the binary relation $\widetilde{R}_2$ defined by:

$$\begin{aligned} \widetilde{R}_2 &= (R_2 \cup (R_1 \cap R_3))\backslash(R_2\backslash(R_1 \cup R_3)) \\ &= (R_1 \cap R_2) \cup (R_2 \cap R_3) \cup (R_1 \cap R_3) \text{ (using De Morgan's law).} \end{aligned}$$

$\widetilde{R}_2$ is distinct from R_2 if $R_2 \notin \mathcal{B}(R_1, R_3)$. It is clear from the second equality and the definition of betweenness that $\widetilde{R}_2 \in \mathcal{B}(R_1, R_3)$, $\widetilde{R}_2 \in \mathcal{B}(R_1, R_2)$ and $\widetilde{R}_2 \in \mathcal{B}(R_2, R_3)$. Using between-additivity, we can write inequality (3) as:

$$\begin{aligned} d(R_1, \widetilde{R}_2) + d(\widetilde{R}_2, R_3) &> d(R_1, \widetilde{R}_2) + d(\widetilde{R}_2, R_2) + d(R_2, \widetilde{R}_2) \\ &\quad + d(\widetilde{R}_2, R_3) \\ &\iff \\ 0 &> 2d(\widetilde{R}_2, R_2) \text{ (using symmetry)} \end{aligned}$$

which is a contradiction to the non-negativity of d.

A.3. Proposition 1

We first show that any function $d : \mathcal{R} \times \mathcal{R} \to \mathbb{R}_+$ that can be written, for any R_1 and $R_2 \in \mathcal{R}$, as:

$$d(R_1, R_2) = \sum_{(x,x') \in R_1 \Delta R_2} \delta(x, x') \tag{4}$$

for some $\delta : X \times X \to \mathbb{R}_+$ satisfying Zero at Identity Only is a between-additive semi-distance as per Definition 3. To show that d satisfies Zero at Identity Only, assume first that $R_1 = R_2$. Then $(R_1\backslash R_2) = (R_2\backslash R_1) = R_1 \Delta R_2 = \varnothing$ so that:

$$d(R_1, R_2) = \sum_{(x,x') \in \varnothing} \delta(x, x') = 0$$

For the other direction, suppose that $d(R_1, R_2) = \sum_{(x,x') \in R_1 \Delta R_2} \delta(x, x') = 0$. Then $R_1 = R_2$ because assuming otherwise would imply, given the reflexivity of both R_1 and R_2, that at least one ordered pair (x, y) with $x \neq y$ belongs to $R_1 \Delta R_2 \neq \varnothing$ and would therefore contradict the assumption that δ satisfies Zero at Identity Only.

We now observe that the symmetry of d is an immediate consequence, given (4), of the fact that $R_1 \Delta R_2 = R_2 \Delta R_1$.

We finally show that d satisfies between-additivity. That is, for any three reflexive binary relations R_1, R_2 and R_3 such that $R_2 \in \mathcal{B}(R_1, R_3)$, we show that $d(R_1, R_2) + d(R_2, R_3) = d(R_1, R_3)$. Using (4), one can write:

$$d(R_1, R_2) + d(R_2, R_3) = \sum_{(x,x') \in R_1 \Delta R_2} \delta(x, x') + \sum_{(y,y') \in R_2 \Delta R_3} \delta(y, y') \tag{5}$$

We observe that $R \vartriangle R' = (R \cup R')\backslash(R \cap R')$ for any two binary relations R and R'. We now prove that the sets $(R_1 \cup R_2)\backslash(R_1 \cap R_2)$ and $(R_2 \cup R_3)\backslash(R_2 \cap R_3)$ are disjoint. Indeed, suppose that $(x, x') \in (R_1 \cup R_2)\backslash(R_1 \cap R_2)$. Then, either (i) $(x, x') \in R_1\backslash R_2$ or (ii) $(x, x') \in R_2\backslash R_1$. In case (i), we know that $(x, x') \notin R_3\backslash R_2$ (by definition of $R_2 \in \mathcal{B}(R_1, R_3)$) . Since by assumption $(x, x') \notin R_2$, one has $(x, x') \notin R_2\backslash R_3$. Hence $(x, x') \notin (R_2\backslash R_3) \cup (R_3\backslash R_2) = (R_2 \cup R_3)\backslash(R_2 \cap R_3)$.

In case (ii), we know by definition that $(x, x') \notin R_3\backslash R_2$ (because $(x, x') \in R_2$) . Since $R_2 \in \mathcal{B}(R_1, R_3)$, one cannot have $(x, x') \in R_2\backslash R_3$ (because $R_2 \subset R_1 \cup R_3$) . Hence any ordered pair in the set $(R_1 \cup R_2)\backslash(R_1 \cap R_2)$ is not in the set $(R_2 \cup R_3)\backslash(R_2 \cap R_3)$ so that the two sets are disjoint. We now show that:

$$[(R_1 \cup R_2)\backslash(R_1 \cap R_2)] \cup [(R_2 \cup R_3)\backslash(R_2 \cap R_3)] = (R_1 \cup R_3)\backslash(R_1 \cap R_3)$$

We first prove that $[(R_1 \cup R_2)\backslash(R_1 \cap R_2)] \cup [(R_2 \cup R_3)\backslash(R_2 \cap R_3)] \subset (R_1 \cup R_3)\backslash(R_1 \cap R_3)$. Consider for this sake any pair of alternatives $(x, x') \in [(R_1 \cup R_2)\backslash(R_1 \cap R_2)] \cup [(R_2 \cup R_3)\backslash(R_2 \cap R_3)]$. Four (non-mutually exclusive) cases are compatible with this consideration:

(i) $(x, x') \in R_1\backslash R_2$
(ii) $(x, x') \in R_2\backslash R_1$
(iii) $(x, x') \in R_2\backslash R_3$
(iv) $(x, x') \in R_3\backslash R_2$.

Consider case (i): Since $R_2 \in \mathcal{B}(R_1, R_3)$, one cannot have $(x, x') \in R_3$ (because this would imply $(x, x') \in R_1 \cap R_3 \subset R_2$, in contradiction of $(x, x') \in R_1\backslash R_2$) . Hence $(x, x') \in R_1\backslash R_3 \subset (R_1 \cup R_3)\backslash(R_1\cap R_3)$. Suppose now that we are in case (ii). By assumption $(x, x') \notin R_1$ and $(x, x') \in R_2 \subset R_1\cup R_3$ (since $R_2 \in \mathcal{B}(R_1, R_3)$) . Hence $(x, x') \in R_3\backslash R_1 \subset (R_1 \cup R_3)\backslash(R_1 \cap R_3)$. For cases (iii) and (iv), we just apply the argument of case (ii) and (i) (respectively) up to permuting R_1 and R_3.

We now prove that $(R_1 \cup R_3)\backslash(R_1 \cap R_3) \subset [(R_1 \cup R_2)\backslash(R_1 \cap R_2)] \cup [(R_2 \cup R_3)\backslash(R_2 \cap R_3)]$. Let $(x, x') \in (R_1 \cup R_3)\backslash(R_1 \cap R_3)$. This means either that $(x, x') \in R_1\backslash R_3$ or that $(x, x') \in R_3\backslash R_1$. In the first case either $(x, x') \in R_2$ (in which case $(x, x') \in R_2\backslash R_3 \subset [(R_1\cup R_2)\backslash(R_1\cap R_2)]\cup[(R_2\cup R_3)\backslash(R_2\cap R_3)]$) or $(x, x') \notin R_2$ (in which case $(x, x') \in R_1 \backslash R_2 \subset [(R_1 \cup R_2) \backslash (R_1 \cap R_2)] \cup [(R_2 \cup R_3) \backslash (R_2 \cap R_3)]$) . The argument for the other case is similar since the sets $(R_1 \cup R_2) \backslash (R_1 \cap R_2)$ and $(R_2 \cup R_3) \backslash (R_2 \cap R_3)$ are disjoint and:

$$[(R_1 \cup R_2) \backslash (R_1 \cap R_2)] \cup [(R_2 \cup R_3) \backslash (R_2 \cap R_3)] = (R_1 \cup R_3) \backslash (R_1 \cap R_3)$$

One can write equality (5) as:

$$\begin{aligned} d(R_1, R_2) + d(R_2, R_3) &= \sum_{(x,x') \in R_1 \triangle R_2} \delta(x, x') + \sum_{(y,y') \in R_2 \triangle R_3} \delta(y, y') \\ &= \sum_{(x,x') \in R_1 \triangle R_3} \delta(x, x') \\ &= d(R_1, R_3) \end{aligned}$$

as required by between-additivity.

In the other direction, we now prove that any between-additive semi-distance function $d : \mathcal{R} \times \mathcal{R} \to \mathbb{R}_+$ can be written as per (4) for some function $\delta : X \times X \to \mathbb{R}_+$ satisfying Zero at Identity Only. We first show that any between-additive distance function $d : \mathcal{R} \times \mathcal{R} \to \mathbb{R}_+$ satisfies the (very strong) independence property that for any two reflexive binary relations R_1 and R_2 and alternatives x and $y \in X$ such that $(x, y) \notin R_1 \cup R_2$, one has:

$$d(R_1, R_1 \cup \{(x, y)\}) = d(R_2, R_2 \cup \{(x, y)\}) \tag{6}$$

Hence, the "distancing" from an existing binary relation brought about by adding to it any ordered pair of alternatives does not depend upon the binary relation to which the ordered pair is added. It only depends on the ordered pair itself. To show this, we first observe that:

$$R_1 \cup \{(x, y)\} \in \mathcal{B}(R_1, R_2 \cup \{(x, y)\}) \tag{7}$$

because:

$$\begin{aligned} R_1 \cap (R_2 \cup \{(x, y)\}) = R_1 \cap R_2 \subset R_1 \subset R_1 \cup \{(x, y)\} \subset R_1 \cup \{(x, y)\} \\ \cup R_2 = R_1 \cup (R_2 \cup \{(x, y)\}) \end{aligned}$$

We also observe that:

$$R_2 \in \mathcal{B}(R_1, R_2 \cup \{(x, y)\}) \tag{8}$$

because:

$$\begin{aligned} R_1 \cap (R_2 \cup \{(x, y)\}) = R_1 \cap R_2 \subset R_2 \subset R_2 \cup \{(x, y)\} \subset R_2 \cup \{(x, y)\} \\ \cup R_1 = R_1 \cup (R_2 \cup \{(x, y)\}) \end{aligned}$$

For the same reason, we also conclude that:

$$R_2 \cup \{(x, y)\} \in \mathcal{B}(R_2, R_1 \cup \{(x, y)\})$$

and:

$$R_1 \in \mathcal{B}(R_2, R_1 \cup \{(x, y)\})$$

It follows from between-additivity that:

$$\begin{aligned} d(R_1, R_2 \cup \{(x, y)\}) &= d(R_1, R_1 \cup \{(x, y)\}) + d(R_1 \cup \{(x, y)\}, R_2 \\ &\quad \cup \{(x, y)\}) \\ &= d(R_1, R_2) + d(R_2, R_2 \cup \{(x, y)\}) \end{aligned}$$

Hence (summing these two last equalities):

$$\begin{aligned} 2d(R_1, R_2 \cup \{(x, y)\}) &= d(R_1, R_1 \\ &\quad \cup \{(x, y)\}) + d(R_1 \cup \{(x, y)\}, R_2 \cup \{(x, y)\}) \\ &\quad + d(R_1, R_2) + d(R_2, R_2 \cup \{(x, y)\}) \\ &= 2d(R_2, R_1 \cup \{(x, y)\}) \end{aligned}$$

thanks to the symmetry of d. Hence:

$$\begin{aligned} d(R_1, R_2 \cup \{(x, y)\}) &= d(R_1, R_1 \cup \{(x, y)\}) + d(R_1 \cup \{(x, y)\}, R_2 \\ &\quad \cup \{(x, y)\}) \\ &= d(R_2, R_1 \cup \{(x, y)\}) \\ &= d(R_2, R_2 \cup \{(x, y)\}) + d(R_2 \cup \{(x, y)\}, R_1 \\ &\quad \cup \{(x, y)\}) \end{aligned}$$

which implies (thanks again to the symmetry of d) that $d(R_1, R_1 \cup \{(x, y)\}) = d(R_2, R_2 \cup \{(x, y)\})$, as required. Define now the function $\delta : X \times X \to \mathbb{R}_+$ for any $(x, y) \in X \times X$ by:

$$\delta(x, y) = d(R, R \cup \{(x, y)\})$$

for some $R \in \mathcal{R}$ such that $(x, y) \notin R$ if $x \neq y$ and by:

$$\delta(x, x) = 0$$

for all $x \in X$. This function δ is well-defined since, as we just established, $d(R_1, R_1 \cup \{(x, y)\}) = d(R_2, R_2 \cup \{(x, y)\}) > 0$ for all R_1 and $R_2 \in \mathcal{R}$ such that $(x, y) \notin (R_1 \cup R_2)$ if $x \neq y$. It satisfies by construction $\delta(x, x) = 0$ for every $x \in X$.

Consider now any two distinct reflexive binary relations R_1 and R_2. Observe that, for any $(x, y) \in R_1 \backslash R_2 = R_1 \backslash (R_1 \cap R_2)$ one has $R_1 \backslash \{(x, y)\} \in \mathcal{B}(R_1, R_2)$.

Therefore, exploiting the between-additivity of d:

$$\begin{aligned} d(R_1, R_2) &= d(R_1, R_1 \backslash \{(x, y)\}) + d(R_1 \backslash \{(x, y)\}, R_2) \\ &= \delta(x, y) + d(R_1 \backslash \{(x, y)\}, R_2) \end{aligned} \tag{9}$$

thanks to the definition of δ and the symmetry of d. Since $R_1 = \bigcup_{(x,y) \in R_1 \backslash R_2} \{(x, y)\} \cup (R_1 \cap R_2)$, one can apply the argument leading to Equality (9) iteratively and obtain:

$$d(R_1, R_2) = \sum_{(x,y) \in R_1 \backslash R_2} \delta(x, y) + d(R_1 \cap R_2, R_2)$$

One can finally apply a similar decomposition of $d(R_1 \cap R_2, R_2)$ (working this time with $R_2 \backslash R_1 = R_2 \backslash (R_1 \cap R_2)$) to finally obtain:

$$d(R_1, R_2) = \sum_{(x,y) \in R_1 \backslash R_2} \delta(x, y) + \sum_{(x',y') \in R_2 \backslash R_1} \delta(x', y')$$

as required by (4).

A.4. Theorem 1

A.4.1. Sufficiency

Suppose that R^* and d are, respectively, a majoritarian binary relation for a profile $(R_1, \ldots, R_n) \in \mathcal{R}^n$ (for some integer $n \geq 2$) and a between-additive semi-distance function from $\mathcal{R} \times \mathcal{R} \to \mathbb{R}_+$ Consider any reflexive binary relation $R \in \mathcal{R}$. We need to show that $\sum_{i=1}^n d(R_i, R^*) \leq \sum_{i=1}^n d(R_i, R)$. Using Proposition 1, this amounts to show that:

$$\begin{aligned} &\sum_{i=1}^n \sum_{(x_i,x_i') \in R_i \Delta R^*} \delta(x_i, x_i') \leq \sum_{i=1}^n \sum_{(y_i,y_i') \in R_i \Delta R} \delta(y_i, y_i') \\ &\iff \\ &\sum_{(x,x') \in X \times X} \delta(x, x') \# \{i : (x, x') \in R_i \Delta R^*\} \\ &\leq \sum_{(x,x') \in X \times X} \delta(x, x') \# \{i : (x, x') \in R_i \Delta R\} \end{aligned} \tag{10}$$

for some function $\delta : X \times X \to \mathbb{R}_+$ satisfying Zero at Identity Only.

By reflexivity of R^*, R_i (for any i) and R, we can assume that all pairs of alternatives (x, x') involved in inequality (10) are such that $x \neq x'$. We will now show that for any pair (x, x'):

$$\# \{i : (x, x') \in R_i \Delta R^*\} \leq \# \{i : (x, x') \in R_i \Delta R\}. \tag{11}$$

By contradiction, suppose there is a pair $(\widehat{x}, \widehat{x}') \in X \times X$ such that:

$$\# \{i : (\widehat{x}, \widehat{x}') \in R_i \Delta R^*\} > \# \{i : (\widehat{x}, \widehat{x}') \in R_i \Delta R\}$$

or, equivalently:

$$\#\{i : (\widehat{x}, \widehat{x}') \in R_i \backslash R^*\} + \#\{i : (\widehat{x}, \widehat{x}') \in R^* \backslash R_i\} > \#\{i : (\widehat{x}, \widehat{x}') \in R_i \backslash R\} + \#\{i : (\widehat{x}, \widehat{x}') \in R \backslash R_i\} \tag{12}$$

By definition of R^* being majoritarian for the profile $(R_1, \ldots, R_n)$, one has (using Remark 1):

$$\#\{i : (\widehat{x}, \widehat{x}') \in R_i \backslash R^*\} \leq n/2$$

and:

$$\#\{i : (\widehat{x}, \widehat{x}') \in R^* \backslash R_i\} < n/2$$

Moreover, one of the four following mutually exclusive possibilities must hold:

(i) $(\widehat{x}, \widehat{x}') \in R^* \backslash R$
(ii) $(\widehat{x}, \widehat{x}') \in R \backslash R^*$
(iii) $(\widehat{x}, \widehat{x}') \notin R \cup R^*$
(iv) $(\widehat{x}, \widehat{x}') \in R \cap R^*$.

In case (i) $\#\{i : (\widehat{x}, \widehat{x}') \in R_i \backslash R^*\} = 0 = \#\{i : (\widehat{x}, \widehat{x}') \in R \backslash R_i\}$. Hence inequality (12) applied to this case reduces to:

$$\begin{aligned} n/2 &> \#\{i : (\widehat{x}, \widehat{x}') \in R^* \backslash R_i\} \\ &> \#\{i : (\widehat{x}, \widehat{x}') \in R_i \backslash R\} \\ &= \#\{i : (\widehat{x}, \widehat{x}') \in R_i \cap R^*\} \\ &\geq n/2 \end{aligned}$$

where the last inequality results from the definition of R^* being majoritarian (Remark 1). This is a contradiction.

In case (ii), one has $\#\{i : (\widehat{x}, \widehat{x}') \in R^* \backslash R_i\} = 0 = \#\{i : (\widehat{x}, \widehat{x}') \in R_i \backslash R\}$. Hence inequality (12) applied to this case writes:

$$\begin{aligned} n/2 &\geq \#\{i : (\widehat{x}, \widehat{x}') \in R_i \backslash R^*\} \\ &> \#\{i : (\widehat{x}, \widehat{x}') \in R \backslash R_i\} \\ &= \#\{i : (\widehat{x}, \widehat{x}') \notin R_i \cup R^*\} \\ &\geq n/2 \end{aligned}$$

which is also a contradiction.

If case (iii) holds, one has $\#\{i : (\widehat{x}, \widehat{x}') \in R^* \backslash R_i\} = 0 = \#\{i : (\widehat{x}, \widehat{x}') \in R \backslash R_i\}$ so that inequality (12) writes:

$$\begin{aligned} n/2 &\geq \#\{i : (\widehat{x}, \widehat{x}') \in R_i \backslash R^*\} \\ &= \#\{i : (\widehat{x}, \widehat{x}') \in R_i\} \\ &> \#\{i : (\widehat{x}, \widehat{x}') \in R_i \backslash R\} \\ &= \#\{i : (\widehat{x}, \widehat{x}') \notin R_i\} \end{aligned}$$

which is again a contradiction.

Finally if one assumes that $(\widehat{x}, \widehat{x}') \in R \cap R^*$, then $\{i : (\widehat{x}, \widehat{x}') \in R_i \backslash R^*\} = \{i : (\widehat{x}, \widehat{x}') \in R_i \backslash R\}$ and $\{i : (\widehat{x}, \widehat{x}') \in R^* \backslash R_i\} = \{i : (\widehat{x}, \widehat{x}') \in R \backslash R_i\}$ so that inequality (12) is contradictory in that case as well.

Hence, for every reflexive binary relation R, a majoritarian R^* satisfies inequality (11) for any pair of alternatives (x, x') and, therefore, satisfies also inequality (10).

A.4.2. Necessity

Let R^* be a reflexive binary relation on X that is not majoritarian for the profile of reflexive binary relations $(R_1, \ldots, R_n)$ for some integer $n \geq 2$. We must show that for any between-additive semi-distance function $d : \mathcal{R} \times \mathcal{R} \to \mathbb{R}_+$, one can find a binary relation R' such that

$$\sum_{i=1}^{n} d(R_i, R^*) > \sum_{i=1}^{n} d(R_i, R') \tag{13}$$

For this sake, we observe that R^* not being majoritarian means (Definition 2) that $R^* \notin \mathcal{B}(R^w(R_1, \ldots, R_n), R^s(R_1, \ldots, R_n))$. Hence either (i) $R^w(R_1, \ldots, R_n) \cap R^s(R_1, \ldots, R_n) \nsubseteq R^*$ or (ii) $R^* \nsubseteq R^w(R_1, \ldots, R_n) \cup R^s(R_1, \ldots, R_n)$. Case (i) implies the existence of alternatives x and $y \in X$ for which $\#\{i : (x, y) \in R_i\} > n/2$ and $(x, y) \notin R^*$. Define for this case the binary relation R' by $R' = R^* \cup \{(x, y)\}$. Consider then any between-additive semi-distance function $d : \mathcal{R} \times \mathcal{R} \to \mathbb{R}_+$. We have, using Proposition 1:

$$\sum_{i=1}^{n} d(R_i, R^*) - \sum_{i=1}^{n} d(R_i, R') = \sum_{i=1}^{n} \Big(\sum_{(x_i, x_i') \in R_i \Delta R^*} \delta(x_i, x_i') - \sum_{(y_i, y_i') \in R_i \Delta R'} \delta(y_i, y_i') \Big) \tag{14}$$

for some function $\delta : X \times X \longrightarrow \mathbb{R}_+$ satisfying Zero at Identity Only.

Equality (14) can equivalently be written as:

$$\begin{aligned} \sum_{i=1}^{n} d(R_i, R^*) - \sum_{i=1}^{n} d(R_i, R') &= \sum_{(a,b) \in X \times X} [\#\{i : (a, b) \in R_i \Delta R^*\} \\ &\quad - \#\{i : (a, b) \in R_i \Delta R'\}]\delta(a, b) \\ &= \sum_{(a,b) \in X \times X : (a,b) \neq (x,y)} [\#\{i : (a, b) \in R_i \Delta R^*\} \\ &\quad - \#\{i : (a, b) \in R_i \Delta R'\}]\delta(a, b) \\ &\quad + [\#\{i : (x, y) \in R_i \Delta R^*\} - \#\{i : (x, y) \in R_i \Delta R'\}]\delta(x, y) \\ &= [\#\{i : (x, y) \in R_i \backslash R^*\} - \#\{i : (x, y) \in (R^* \cup \{(x, y)\}) \backslash R_i\}] \\ &\quad \times \delta(x, y) \\ &> [n/2 - \#\{i : (x, y) \in (R^* \cup \{(x, y)\}) \backslash R_i\}]\delta(x, y) \\ &\geq 0 \text{ (because } n/2 \geq \#\{i : (x, y) \in (R^* \cup \{(x, y)\}) \backslash R_i\}) \end{aligned}$$

which establishes inequality (13) for this case. Case (ii) on the other hand implies the existence of alternatives x and $y \in X$ for which $\#\{i : (x, y) \in R_i\} < n/2$ and $(x, y) \in R^*$. Define for this case the binary relation R' by $R' = R^* \backslash \{(x, y)\}$.

One would then have (following the same reasoning as above):

$$\begin{aligned} \sum_{i=1}^{n} d(R_i, R^*) - \sum_{i=1}^{n} d(R_i, R') &= \sum_{i=1}^{n} \Big(\sum_{(x_i, x_i') \in R_i \Delta R^*} \delta(x_i, x_i') - \sum_{(y_i, y_i') \in R_i \Delta R'} \delta(y_i, y_i') \Big) \\ &= [\#\{i : (x, y) \in R^* \backslash R_i\} - \#\{i : (x, y) \in R_i \backslash (R^* \backslash \{(x, y)\})\}]\delta(x, y) \\ &> [n/2 - \#\{i : (x, y) \in R_i \backslash (R^* \backslash \{(x, y)\})\}]\delta(x, y) \\ &> 0 \text{ (because } n/2 > \#\{i : (x, y) \in R_i \backslash (R^* \backslash \{(x, y)\})\}) \end{aligned}$$

A.5. Theorem 2

Suppose that a distance $d : \mathcal{R} \times \mathcal{R} \to \mathbb{R}_+$ is not between-additive. This means that there are reflexive binary relations R_1, R_2 and R_3 on X such that $R_2 \in \mathcal{B}(R_1, R_3)$ and:

$$d(R_1, R_3) < d(R_1, R_2) + d(R_2, R_3) \tag{15}$$

using the Triangle Inequality. We will show that, for any even integer n, one can construct a profile of reflexive binary relations with respect to which a majoritarian binary relation will not minimize the sum of distances between itself and the binary relations of the profile. We will also show, somewhat less strongly, that for infinitely many odd integers (but not necessarily for all of them), we can also make such constructions. Suppose first that n is even. Consider the profile $(\widehat{R}_1, \ldots, \widehat{R}_n) \in \mathcal{R}$ defined by,

$$(\widehat{R}_1, \ldots, \widehat{R}_n) = (\underbrace{R_1, \ldots, R_1}_{n/2}, \underbrace{R_3, \ldots, R_3}_{n/2})$$

It is clear that the two binary relations R_1 and R_3 are majoritarian on this profile. Let us now show that R_2 is also majoritarian on this profile. For this sake we use Remark 1, and we first consider any alternatives x and y such that $(x, y) \in R_2$.

Since $R_2 \in \mathcal{B}(R_1, R_3)$, one must have $(x, y) \in R_1 \cup R_3$. Hence, at least one of the two binary relations R_1, or R_3 must contain the pair (x, y) . Hence $\#\{i : (x, y) \in \widehat{R}_i\} \geq \frac{n}{2}$.

Consider now x and y such that $(x, y) \notin R_2$. Since $R_2 \in \mathcal{B}(R_1, R_3)$ and, as a result, $R_1 \cap R_3 \subset R_2$, one must have that $(x, y) \notin R_1 \cap R_3$. Hence there can be at most one of the two binary relations R_1 and R_3 that contain the pair (x, y). Put differently $\#\{i : (x, y) \in \widehat{R}_i\} \leq \frac{n}{2}$ as required by the second condition of Remark 1. Hence R_2 is indeed majoritarian on the profile $(\widehat{R}_1, \ldots, \widehat{R}_n)$. However, R_2 does not minimize the sum of distances between itself and each of the n binary relations of the profile because,from inequality (15) and the property of Zero at Identity Only, one has (using Symmetry):

$$\sum_{i=1}^{n} d(R_1, \widehat{R}_i)$$
$$= (n/2)d(R_1, R_1) + (n/2)d(R_1, R_3)$$
$$= (n/2)d(R_1, R_3)$$
$$< (n/2)d(R_1, R_2) + (n/2)d(R_2, R_3)$$
$$= \sum_{i=1}^{n} d(R_2, \widehat{R}_i)$$

Hence R_1 (but the argument would work just as well for R_3) has a strictly smaller aggregate distance from the binary relations of the profile than R_2. Suppose now we focus on profiles with an odd number n of binary relations. For any such n, consider the profile $(\widehat{R}_1, \ldots, \widehat{R}_n)$ defined by:

$$(\widehat{R}_1, \ldots, \widehat{R}_n) = (\underbrace{R_1, \ldots, R_1}_{(n-1)/2}, R_2, \underbrace{R_3, \ldots, R_3}_{(n-1)/2})$$

It is easy to see that R_2 is the unique majoritarian binary relation on any such profile. Let us show that there exists some odd $n^* \geq 3$ such that for any $n \geq n^*$, one has:

$$\sum_{i=1}^{n} d(R_i^*, \widehat{R}_i) < \sum_{i=1}^{n} d(R_2, \widehat{R}_i)$$

for some binary relation R_i^* distinct from R_2 (so that the majoritarian R_2 does not minimize the sum of distances between itself and all other binary relations of the profile for such n) . Define for this purpose $R_i^* \in \arg\min_{R_i \in \{R_1, R_3\}} d(R_i, R_2)$ and suppose, by contradiction, that for all odd n, one has:

$$\sum_{i=1}^{n} d(R_i^*, \widehat{R}_i) \geq \sum_{i=1}^{n} d(R_2, \widehat{R}_i)$$

Given the definition of the profile $(\widehat{R}_1, \ldots, \widehat{R}_n)$, this is equivalent to assuming that for all odd n:

$$\frac{(n-1)}{2} d(R_i^*, R_1) + d(R_i^*, R_2) + \frac{(n-1)}{2} d(R_i^*, R_3)$$
$$= d(R_i^*, R_2) + \frac{(n-1)}{2} d(R_1, R_3)$$
$$\geq \frac{(n-1)}{2} d(R_1, R_2) + \frac{(n-1)}{2} d(R_2, R_3)$$
$$\Longleftrightarrow$$
$$d(R_i^*, R_2) \geq \frac{(n-1)}{2}[d(R_1, R_2) + d(R_2, R_3) - d(R_1, R_3)]$$

where the first equality uses the property of Zero at Identity Only. But observing this inequality for any odd integer n is incompatible with the (positive) finiteness of $d(R_i^*, R_2) = \min(d(R_1, R_2), d(R_2, R_3))$ and the assumed positivity of $d(R_1, R_2) + d(R_2, R_3) - d(R_1, R_3)$.

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